Axisymmetric free convection boundary layer flow of water at 4°C past slender bodies

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The transverse curvature effects on axisymmetric free convection boundary layer flow of water at 4°C past a vertical cylinder were investigated. The governing equations for momentum and energy were solved numerically. Missing values of the velocity and thermal functions, which are proportional to the surface friction and the dimensionless heat transfer rate, were tabulated for a Prandtl number of 11.4. It was observed that the heat transfer rate increses with increasing transverse curvature.

Keywords: *free convection flow, boundary layer flow, numerical methods, heat transfer, hydrodynamics*

Introduction

The presence of a buoyancy force is a requirement for the existence of a free convection flow. The buoyancy arises from density differences which are a consequence of the temperature gradients within the fluid. Buoyancy effects in natural convection flows have been extensively investigated[1-5]. In all these studies, the fluids considered were at normal temperature of 20°C. Under these conditions, the variation in density is linear with respect to temperature and is given by $\Delta\rho=\rho\beta\ (T_w-T_\infty)$. But when the water is at 4°C, its density is maximum and the variation of density can be adequately represented[6] by $\Delta\rho=\rho\gamma\ (T_w-T_\infty)^2$ where $\gamma=8\times10^{-6}(°C)^{-2}$. The problem of free and combined convective flow of water at 4°C over a semi-infinite flat plate has been studied[7-10].

The present work was undertaken to study the influence of transverse curvature on laminar free convective boundary layer flow of water at 4°C. To this end, we singled out the simplest body of transverse curvature, namely, a thin circular cylinder.

It is known that for small temperature variations, free convection in water at 4°C is much reduced from that at 20°C. The measurement of the terminal velocity of small particles, working at 4°C, may be of benefit in measuring molecular diffusivities in water and in heat and mass transfer experiments where one would like to suppress natural convection. The rates of heat transfer by free convection in water at 4°C may be reduced considerably from those at other temperatures, and this is an important consideration in certain freezing processes.

Governing equations

Consideration is given here to an axisymmetric, steady, laminar, free convection boundary layer flow of water at 4°C past a thin, vertical, circular cylinder. Fig 1 shows the flow model and the coordinate system used. Under the usual Boussinesq approximation, flow of water at 4°C may be expressed within boundary layer approximation as follows.

Mass:

$$\frac{\partial}{\partial x}(\Omega u)+\frac{\partial}{\partial y}(\Omega V)=0 \tag{1}$$

Momentum:

$$u\frac{\partial u}{\partial x}+V\frac{\partial u}{\partial \Omega}=\frac{v}{\Omega}\frac{\partial}{\partial \Omega}\left(\Omega\frac{\partial u}{\partial \Omega}\right)+\gamma g(T-T_\infty)^2 \tag{2}$$

Energy:

$$u\frac{\partial T}{\partial x}+V\frac{\partial T}{\partial \Omega}=\alpha\frac{1}{\Omega}\frac{\partial}{\partial \Omega}\left(\Omega\frac{\partial T}{\partial \Omega}\right) \tag{3}$$

The boundary conditions are given by

$$\Omega=R:\ u=V=0,\qquad T=T_w(\gtrless 4°C)$$
$$\Omega\to\infty:\ u\to0,\qquad T\to T_\infty(=4°C) \tag{4}$$

In the above equations, u and v are the velocity components in the x and r directions, respectively, T the temperature, g the gravitational acceleration, and α the thermal diffusivity.

Analysis

The usual way of solving the equation of conservation of mass is to define a stream function $\psi(x,r)$ such that

$$u=\frac{1}{\Omega}\frac{\partial\psi}{\partial\Omega}$$
$$V=-\frac{1}{\Omega}\frac{\partial\psi}{\partial x} \tag{5}$$

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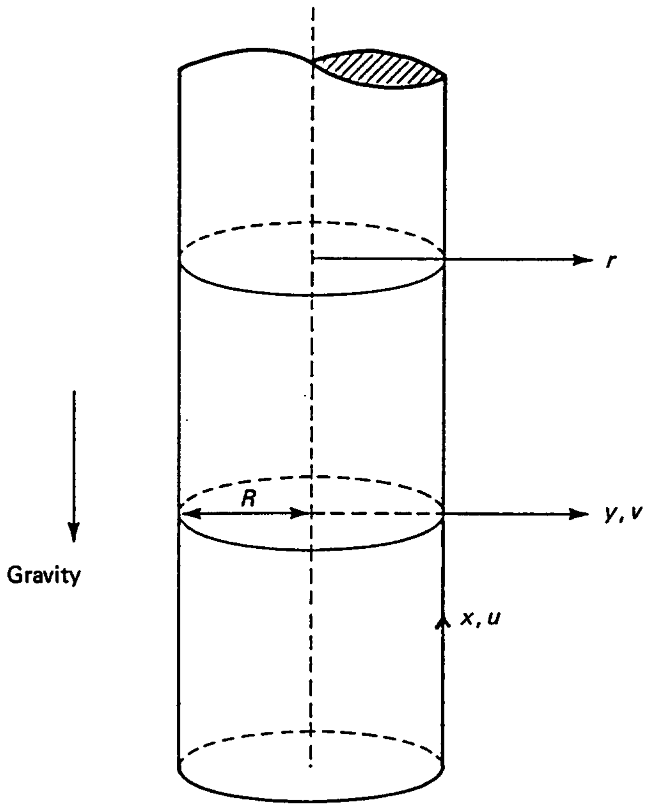

Fig 1 Coordinate system and flow model

We further define

$$\xi = \frac{2L}{CR}(\bar{x})^{1/4}$$
$$\eta = \frac{\Omega^2 R^2}{R^2 \xi}$$
$$C = \frac{g\gamma(T_w - T_\infty)L^3}{4\nu^2} \quad (6)$$
$$\psi = 4\nu RC(\bar{x})^{3/4} f(\xi, \eta)$$
$$\theta = \frac{T - T_\infty}{T_w - T_\infty}$$
$$\bar{x} = (x/L)$$

After substituting the expressions in (6) into Eqs (2) and (3), we obtain

$$(1+\xi\eta)f''' + 3ff'' - 2(f')^2 + \theta^2 + \xi \times \left[f'' - f'\frac{\partial^2 f}{\partial\eta\partial\xi} + f''\frac{\partial f}{\partial\xi}\right] = 0 \quad (7)$$

$$\frac{1}{Pr}(1+\xi\eta)\theta'' + 3f\theta' + \xi\left[\frac{1}{\rho\Omega}\theta' - f'\frac{\partial\theta}{\partial\xi} + \theta'\frac{\partial f}{\partial\xi}\right] = 0 \quad (8)$$

The transformed boundary conditions are given by

$$f(\xi,0) = f'(\xi,0), \qquad \theta(\xi,0) = 1$$
$$f'(\xi,\infty) = \theta(\xi,\infty) = 0 \quad (9)$$

In the above equations, a prime denotes differentiation with respect to η only.

In order to solve the partial differential equations (7) and (8), we assume that

$$f(\xi,\eta) = f_0(\eta) + \xi f_1(\eta) + \xi^2 f_2(\eta) + \ldots \quad (10)$$
$$\theta(\xi,\eta) = \theta_0(\eta) + \xi\theta_1(\eta) + \xi^2\theta_2(\eta) + \ldots \quad (11)$$

When Eqs (10) and (11) are substituted into Eqs (7) and (8), and terms involving equal powers of ξ equated to zero, one obtains the following set of ordinary differential equations governing the momentum and energy fields:

$$f_0''' + 3f_0 f_0'' - 2(f_0')^2 + \theta_0^2 = 0 \quad (12)$$

$$\frac{\theta_0''}{Pr} + 3f_0\theta_0^1 = 0 \quad (13)$$

$$f_1''' + \eta f_0''' + 3f_0 f_1'' + 4f_1 f_0'' + f_0'' - 5f_0' f_0' + 2\theta_0\theta_1 = 0 \quad (14)$$

$$\frac{\theta_1''}{Pr} + \frac{\eta\theta_0''}{Pr} + \frac{\theta_0'}{Pr} + 3f_0\theta_0' + 4f_1\theta_0' - \theta_1 f_0' = 0 \quad (15)$$

$$f_2''' + \eta f_1''' + 3f_0 f_2'' + 4f_1 f_1'' + 5f_2 f_0'' + f_1'' - 6f_0' f_2' - 3(f_1')^2 + 2\theta_0\theta_2 + \theta_1^2 = 0 \quad (16)$$

$$\frac{\theta_2''}{Pr} + \frac{\eta\theta_1''}{Pr} + \frac{\theta_1'}{Pr} + 3f_0\theta_2' + 4f_1\theta_1' + 5f_2\theta_0' - \theta_1 f_1' - 2\theta_2 f_0' = 0 \quad (17)$$

$$f_3''' + \eta f_2''' + f_2'' + 3f_0 f_3'' + 4f_1 f_2'' + 5f_2 f_1'' - 7f_0' f_3' - 7f_1' f_2' + 6f_0'' f_3 + 2\theta_0\theta_3 + 2\theta_1\theta_2 = 0 \quad (18)$$

Notation

C	Dimensionless number
C_p	Specific heat at constant pressure
f	Dependent variable representing the stream function
g	Gravitational acceleration
Gr_x	Local Grashof number
k	Thermal conductivity
L	Characteristic length
Nu	Local Nusselt number
Pr	Prandtl number
q	Heat flux
r	Radial coordinate
R	Radius of cylinder
T	Temperature
u	Velocity component in x direction
v	Velocity component in r direction
x	Longitudinal coordinate
β	Coefficient of thermal expansion
η	Dimensionless coordinate
θ	Dimensionless temperature
ν	Kinematic viscosity
ξ	Dimensionless coordinate
ρ	Density of the fluid
τ	Shear stress
ψ	Stream function

Subscripts

∞	Ambient conditions
w	Wall conditions

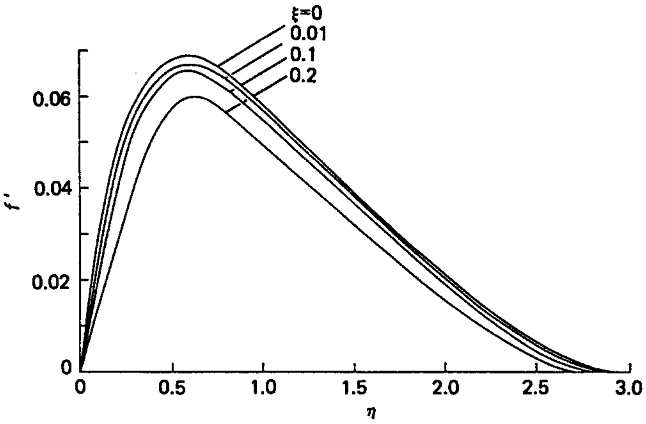

Fig 2 Distribution of velocity profiles

$$\frac{\theta_3''}{Pr}+\frac{\eta\theta_2''}{Pr}+\frac{\theta_2'}{Pr}+3f_0\theta_3'+4f_1\theta_2'+5f_2\theta_1'+6f_3\theta_0'-\theta_1 f_2'-2\theta_2 f_2'-3\theta_3 f_0'=0 \quad (19)$$

$$f_4'''+\eta f_3'''+f_3''+3f_0f_4''+4f_1f_3''+5f_2f_2''+6f_3f_1''+7f_4f_0''-8f_0'f_4'-8f_1'f_3'-4(f_2')^2+2\theta_0\theta_4+2\theta_1\theta_3+(\theta_2)^2=0 \quad (20)$$

$$\frac{\theta_4''}{Pr}+\frac{\eta\theta_3''}{Pr}+\frac{\theta_3'}{Pr}+3f_0\theta_4'+4f_1\theta_3'+5f_2\theta_2'+6f_3\theta_1'+7f_4\theta_0'-\theta_1 f_3'-2\theta_2 f_2'-3\theta_3 f_1'-4\theta_4 f_0'=0 \quad (21)$$

etc.

The appropriate boundary conditions are given by

$$f_j(0)=f_j'(0)=f_j'(\infty)=0 \quad \text{for } j\geqslant 0 \quad (22)$$

$$\theta_0(0)=1,\ \theta_j(0)=0 \quad \text{for } j\geqslant 1 \quad (23)$$

$$\theta_j(\infty)=0 \quad \text{for } j\geqslant 0$$

Eqs (12) to (21) were solved numerically on the computer using a fourth-order Runge–Kutta procedure with Pr as a parameter. Double precision arithmetic was used in all the computations. A step size of $\Delta\eta=0.001$ was selected. The missing wall values for the velocity and thermal functions were determined by shooting techniques.

The wall shear stress may be written as

$$\tau_w=\left(\mu\frac{\partial u}{\partial\Omega}\right)_{\Omega=R} \quad (24)$$

The local friction coefficient is then given by

$$C_{f_x}=\frac{16C}{\xi^2}(\bar{x})^{3/4}\frac{1}{Re^2}\{f''(\xi,0)\} \quad (25)$$

The local heat transfer from the surface of the cylinder to the fluid is given by Fourier's law:

$$q_w=-K\left(\frac{\partial T}{\partial\Omega}\right)_{\Omega=R}=-\frac{KC(T_w-T_\infty)}{l}(\bar{x})^{-1/4}\theta'(\xi,0) \quad (26)$$

The local Nusselt number is given by

$$\frac{Nu_x}{Gr_x^{1/4}}=-(4)^{1/4}\sum_{n=0}^{\infty}\xi^n\theta_n'(0) \quad (27)$$

The velocity and temperature profiles are plotted in Figs 2 and 3. Here the Prandtl number Pr has been taken as 11.4 for water at 4°C. The results indicate that the maximum velocity decreases as ξ increases. This may be explained by the fact that smaller temperature differences give rise to smaller velocities. The velocity and temperature boundary layers flatten out less rapidly as ξ becomes larger.

In order to facilitate the calculation of friction factor and the Nusselt number, we tabulated $f_i''(0)$ and $\theta_i'(0)$, as shown in Tables 1 and 2, for $Pr=11.4$. The numerical results indicate that the friction factor (C_{f_x}) and the heat transfer (Nu_x) rate are augmented as the transverse curvature parameter increases. The surface friction coefficient and the wall heat transfer rate are usually of practical importance, and these quantities may be readily evaluated from the information supplied in Tables 1 and 2.

Concluding remarks

The natural convective flow of water at 4°C past a thin vertical cylinder was studied. It was found that, for water at 4°C, the buoyancy effect is not a linear function of the temperature difference but is proportional to the square of the temperature difference. Numerical solutions were presented for the fluid flow and heat transfer characteristics. The missing wall values of the velocity and thermal functions were tabulated for a Prandtl

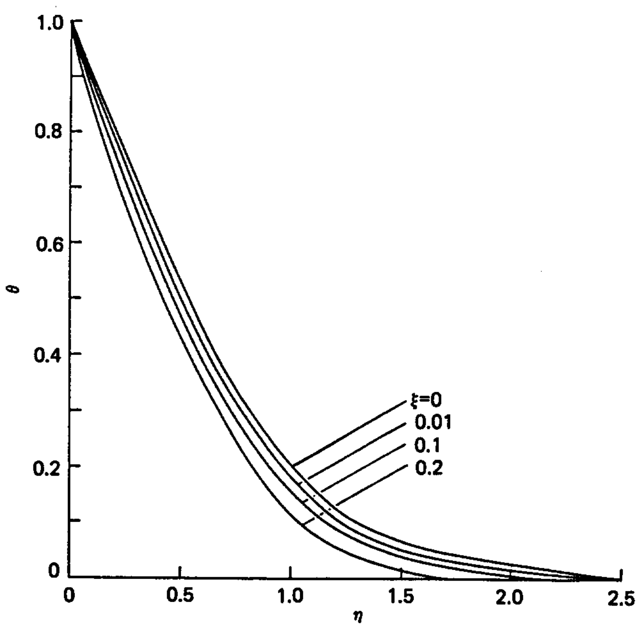

Fig 3 Distribution of temperature profiles

Table 1 Values of $f_i''(0)$ for $Pr=11.4$

$f_0''(0)=$	0.309 51
$f_1''(0)=$	−0.003 992 8
$f_2''(0)=$	0.004 226 7
$f_3''(0)=$	−0.004 277 9
$f_4''(0)=$	0.005 947 9

Table 2 Values of $\theta_i'(0)$ for Pr=11.4

$\theta_0'(0) = -1.005\,76$
$\theta_1'(0) = -0.153\,68$
$\theta_2'(0) = -0.020\,560$
$\theta_3'(0) = 0.011\,333$
$\theta_4'(0) = -0.009\,752\,6$

number of 11.4. This information would be useful for the evaluation of the surface friction factor as well as the heat transfer rate. The numerical results demonstrated that the heat transfer rate increases with increasing transverse curvature.

References

1. **Sparrow, E. M. and Gregg, J. L.** Similar solutions for free convection from a non-isothermal vertical plate. *Trans. ASME, 1958*, **80**, *379*
2. **Finston, M.** Free convection past a vertical plate. *Z. Angew. Math. Phys., 1956*, **17**, *527*
3. **Yang, K. T.** Possible similarity solutions for laminar free convection on vertical plates and cylinders. *J. Appl. Mech., 1960*, **26**, *230*
4. **Sparrow, E. M. and Gregg, J. L.** The laminar free convection heat transfer from the outer surface of a vertical cylinder. *Trans. ASME, 1956*, **78**, *1823*
5. **Millsaps, K. abd Pohlausen, K.** The laminar free convective heat transfer from the outer surface of a vertical circular cylinder. *J. Aeronauti. Sci., 1958*, **25**, *357*
6. **Goren, S. L.** On free convection in water at 4°C. *Chem. Eng. Sci., 1966*, **21**, *515*
7. **Soundalgekar, V. M.** Laminar free convection flow of water at 4°C from a vertical plate with variable wall temperature. *Chem. Eng. Sci., 1973*, **28**, *307*
8. **Gebhart, B. and Mollendorf, J. C.** A new density relation for pure and saline water. *Deep Sea Res., 1977*, **24**, *831*
9. **Gebhart, B. and Mollendorf, J. C.** Buoyancy induced flows in water under conditions in which density extrema may arise. *J. Fluid Mech., 1978*, **89**, *673*
10. **Soundalgekar, V. M., Ramana Murty, T. V. and Vighnesam, N. V.** Combined forced and free convective flow of water at 4°C past a semi-infinite vertical plate. *Int. J. Heat and Fluid Flow, 1984*, **5**, *54*

BOOK REVIEW

Fundamentals of Flow Measurement

J. P. DeCarlo

The International Society of America has published a number of ILM's (Independent Learning Modules) on control principles and techniques, fundamental instrumentation and unit process and unit operational control. The ILM's aim to be self-contained books which allow the reader to teach himself the subjects covered by the series of modules.

Clearly the authors have to start with the assumption that their readers have a broad understanding of basic physics and engineering. There is thus no attempt to give detailed or mathematical expositions but it is hoped that sufficient explanation is there to allow readers to grasp the essential characteristics.

Joseph DeCarlo achieves this goal excellently in this ILM, Fundaments of Flow Measurement, though inevitably in trying to cover such a wide field there have to be many areas where only the surface is scratched. The most serious gap is on the assessment of uncertainties for although several pages deal with the terminology of random and systematic uncertainties, etc, no reference is made to the ISO Standard 5168 and the examples are not adequate to enable the reader to carry out his own assessment.

The format is good, with the book being divided into 12 units, each with a simple set of objectives and a summary and a series of exercises at the end. The first three units deal with the general background of classification and terminology and the next eight with the different groups of closed conduit and open channel devices and techniques.

The final unit deals with flowmeter selection. This is a most difficult subject and in the example used to illustrate the method their are a number of instances where the choice is not as straightforward as the author would think. Nevertheless the author's choice of the phrase 'smiles per dollar' adds a moment of lightness to a very serious subject.

Overall this book fulfils a real need for there are very few books which provide a general introduction to the whole subject. Although it is oriented to use in the USA, and the references are mainly American, this should not deter readers in other countries. An Appendix giving well-presented solutions to the exercises in each unit is admirable.

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